

ON MULTIPLIERS OF HILBERT MODULES OVER LOCALLY C^* -ALGEBRAS

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ABSTRACT. In this paper, we investigate the structure of the multiplier module of a Hilbert module over a locally C^* -algebra and the relationship between the set of all adjointable operators from a Hilbert A -module E to a Hilbert A -module F and the set of all adjointable operators from the multiplier module $M(E)$ of E to the multiplier module $M(F)$ of F .

1. INTRODUCTION

Locally C^* -algebras are generalizations of C^* -algebras. Instead of being given by a single norm, the topology on a locally C^* -algebra is defined by a directed family of C^* -seminorms. Such important concepts as multiplier algebra, Hilbert C^* -module, adjointable operator, multiplier module of a Hilbert C^* -module can be defined in the framework of locally C^* -algebras.

In this paper, we investigate the multipliers of Hilbert modules over locally C^* -algebras. A multiplier of a Hilbert A -module E is an adjointable operator from A to E . The set $M(E)$ of all multipliers of E is a Hilbert $M(A)$ -module in a natural way. We show that $M(E)$ is an inverse limit of multiplier modules of Hilbert C^* -modules and E can be identified with a closed submodule of $M(E)$ which is strictly dense in $M(E)$ (Theorem 3.3). For a countable set $\{E_n\}_n$ of Hilbert A -modules, the multiplier module $M(\bigoplus_n E_n)$ of $\bigoplus_n E_n$ can be identified with the set of all sequences $(t_n)_n$ with $t_n \in M(E_n)$ such that $\sum_n t_n^* \circ t_n$ converges strictly in $M(A)$ (Theorem 3.9). This is a generalization of a result of Bakic and Guljas [1] which states that $M(H_A)$ (H_A is the Hilbert C^* -module of all sequences $(a_n)_n$ in A

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such that $\sum_n a_n^* a_n$ converges in the C^* -algebra A) is the set of all sequences $(m_n)_n$ in $M(A)$ such that the series $\sum_n m_n^* m_n a$ and $\sum_n a m_n^* m_n$ converge in A for all a in A .

Section 4 is devoted to study the connection between the set of all adjointable operators between two Hilbert A -modules E and F , and the set of all adjointable operators between the multiplier modules $M(E)$ and $M(F)$. Given two Hilbert A -modules E and F , we show that any adjointable operator from $M(E)$ to $M(F)$ is strictly continuous and the locally convex space $L_A(E, F)$ of all adjointable operators from E to F is isomorphic with the locally convex space $L_{M(A)}(M(E), M(F))$ of all adjointable operators from $M(E)$ to $M(F)$ (Theorem 4.1). In particular the locally C^* -algebras $L_A(E)$ and $L_{M(A)}(M(E))$ are isomorphic. The last result is a generalization of a result of Bakic and Guljas [1] which states that the C^* -algebra of all adjointable operators on a full Hilbert C^* -module is isomorphic with the C^* -algebra of all adjointable operators on the multiplier module. Also we show that two Hilbert modules E and F are unitarily equivalent if and only if the multiplier modules $M(E)$ and $M(F)$ are unitarily equivalent (Corollary 4.2).

2. PRELIMINARIES

A locally C^* -algebra is a complete Hausdorff complex topological $*$ -algebra A whose topology is determined by its continuous C^* -seminorms in the sense that a net $\{a_i\}_{i \in I}$ converges to 0 in A if and only if the net $\{p(a_i)\}_i$ converges to 0 for all continuous C^* -seminorm p on A .

Here, we recall some facts about locally C^* -algebras from [2, 3, 6, 8]. Let A be a locally C^* -algebra.

A multiplier on A is a pair (l, r) of linear maps from A to A such that: $l(ab) = l(a)b$, $r(ab) = ar(b)$ and $al(b) = r(a)b$ for all $a, b \in A$. The set $M(A)$ of all multipliers of A is a locally C^* -algebra with respect to the topology determined by the family of C^* -seminorms $\{p_{M(A)}\}_{p \in S(A)}$, where $p_{M(A)}(l, r) = \sup\{p(l(a)); p(a) \leq 1\}$.

An approximate unit for A is an increasing net of positive elements $\{e_i\}_{i \in I}$ in A such that $p(e_i) \leq 1$ for all $p \in S(A)$ and for all $i \in I$, and $p(ae_i - a) + p(e_i a - a) \rightarrow 0$ for all $p \in S(A)$ and for all $a \in A$. Any locally C^* -algebra has an approximate unit.

An element $a \in A$ is bounded if $\|a\|_\infty = \sup\{p(a); p \in A\} < \infty$. The set $b(A)$ of all bounded elements in A is dense in A and it is a C^* -algebra in the C^* -norm $\|\cdot\|_\infty$.

A morphism of locally C^* -algebras is a continuous morphism of $*$ -algebras. Two locally C^* -algebras A and B are isomorphic if there is a bijective map $\Phi : A \rightarrow B$ such that Φ and Φ^{-1} are morphisms of locally C^* -algebras.

The set $S(A)$ of all continuous C^* -seminorms on A is directed with the order $p \geq q$ if $p(a) \geq q(a)$ for all $a \in A$. For each $p \in S(A)$, $\ker p = \{a \in A; p(a) = 0\}$ is a two-sided $*$ -ideal of A and the quotient algebra $A/\ker p$, denoted by A_p , is a C^* -algebra in the C^* -norm induced by p . The canonical map from A to A_p is denoted by π_p . For $p, q \in S(A)$ with $p \geq q$ there is a canonical surjective morphism of C^* -algebras $\pi_{pq} : A_p \rightarrow A_q$ such that $\pi_{pq}(\pi_p(a)) = \pi_q(a)$ for all $a \in A$ and which extends to a morphism of C^* -algebras $\pi''_{pq} : M(A_p) \rightarrow M(A_q)$. Then $\{A_p; \pi_{pq}\}_{p, q \in S(A), p \geq q}$ is an inverse system of C^* -algebras as well as $\{M(A_p); \pi''_{pq}\}_{p, q \in S(A), p \geq q}$ and moreover, the locally C^* -algebras A and $\varprojlim_p A_p$ are isomorphic as well as $M(A)$ and $\varprojlim_p M(A_p)$.

Hilbert modules over locally C^* -algebras are generalizations of Hilbert C^* -modules [7] by allowing the inner-product to take values in a locally C^* -algebra rather than in a C^* -algebra. Here, we recall some facts about Hilbert modules over locally C^* -algebras from [4, 5, 6, 8].

Definition 1. A pre-Hilbert A -module is a complex vector space E which is also a right A -module, compatible with the complex algebra structure, equipped with an A -valued inner product $\langle \cdot, \cdot \rangle : E \times E \rightarrow A$ which is \mathbb{C} - and A -linear in its second variable and satisfies the following relations:

- (1) $\langle \xi, \eta \rangle^* = \langle \eta, \xi \rangle$ for every $\xi, \eta \in E$;
- (2) $\langle \xi, \xi \rangle \geq 0$ for every $\xi \in E$;
- (3) $\langle \xi, \xi \rangle = 0$ if and only if $\xi = 0$.

We say that E is a Hilbert A -module if E is complete with respect to the topology determined by the family of seminorms $\{\overline{p}_E\}_{p \in S(A)}$ where $\overline{p}_E(\xi) = \sqrt{p(\langle \xi, \xi \rangle)}$, $\xi \in E$.

An element $\xi \in E$ is bounded if $\sup\{\overline{p}_E(\xi); p \in A\} < \infty$. The set $b(E)$ of all bounded elements in E is a Hilbert $b(A)$ -module which is dense in E .

Any locally C^* -algebra A is a Hilbert A -module in a natural way.

A Hilbert A -module E is full if the linear space $\langle E, E \rangle$ generated by $\{\langle \xi, \eta \rangle, \xi, \eta \in E\}$ is dense in A .

Let E be a Hilbert A -module. For $p \in S(A)$, $\ker \bar{p}_E = \{\xi \in E; \bar{p}_E(\xi) = 0\}$ is a closed submodule of E and $E_p = E / \ker \bar{p}_E$ is a Hilbert A_p -module with $(\xi + \ker \bar{p}_E)\pi_p(a) = \xi a + \ker \bar{p}_E$ and $\langle \xi + \ker \bar{p}_E, \eta + \ker \bar{p}_E \rangle = \pi_p(\langle \xi, \eta \rangle)$. The canonical map from E onto E_p is denoted by σ_p^E . For $p, q \in S(A)$, $p \geq q$ there is a canonical morphism of vector spaces σ_{pq}^E from E_p onto E_q such that $\sigma_{pq}^E(\sigma_p^E(\xi)) = \sigma_q^E(\xi)$, $\xi \in E$. Then $\{E_p; A_p; \sigma_{pq}^E, \pi_{pq}\}_{p, q \in S(A), p \geq q}$ is an inverse system of Hilbert C^* -modules in the following sense: $\sigma_{pq}^E(\xi_p a_p) = \sigma_{pq}^E(\xi_p)\pi_{pq}(a_p)$, $\xi_p \in E_p, a_p \in A_p$; $\langle \sigma_{pq}^E(\xi_p), \sigma_{pq}^E(\eta_p) \rangle = \pi_{pq}(\langle \xi_p, \eta_p \rangle)$, $\xi_p, \eta_p \in E_p$; $\sigma_{pp}^E(\xi_p) = \xi_p$, $\xi_p \in E_p$ and $\sigma_{qr}^E \circ \sigma_{pq}^E = \sigma_{pr}^E$ if $p \geq q \geq r$, and $\lim_{\leftarrow p} E_p$ is a Hilbert A -module which can be identified with E .

We say that an A -module morphism $T : E \rightarrow F$ is adjointable if there is an A -module morphism $T^* : F \rightarrow E$ such that $\langle T\xi, \eta \rangle = \langle \xi, T^*\eta \rangle$ for every $\xi \in E$ and $\eta \in F$. Any adjointable A -module morphism is continuous. The set $L_A(E, F)$ of all adjointable A -module morphisms from E into F is a complete locally convex space with topology defined by the family of seminorms $\{\tilde{p}_{L_A(E, F)}\}_{p \in S(A)}$, where $\tilde{p}_{L_A(E, F)}(T) = \|(\pi_p^{E, F})_*(T)\|_{L_{A_p}(E_p, F_p)}$, $T \in L_A(E, F)$ and $(\pi_p^{E, F})_*(T)(\sigma_p^E(\xi)) = \sigma_p^F(T\xi)$, $\xi \in E$. Moreover, $\{L_{A_p}(E_p, F_p); (\pi_{pq}^{E, F})_*\}_{p, q \in S(A), p \geq q}$, where $(\pi_{pq}^{E, F})_* : L_{A_p}(E_p, F_p) \rightarrow L_{A_q}(E_q, F_q)$, $(\pi_{pq}^{E, F})_*(T_p)(\sigma_p^E(\xi)) = \sigma_p^F(T_p(\sigma_p^E(\xi)))$, is an inverse system of Banach spaces, and $\lim_{\leftarrow p} L_{A_p}(E_p, F_p)$ can be identified with $L_A(E, F)$. Thus topologized, $L_A(E, E)$ becomes a locally C^* -algebra, and we write $L_A(E)$ for $L_A(E, E)$.

An element T in $L_A(E, F)$ is bounded in $L_A(E, F)$ if $\|T\|_\infty = \sup\{\tilde{p}_{L_A(E, F)}(T); p \in S(A)\} < \infty$. The set $b(L_A(E, F))$ of all bounded elements in $L_A(E, F)$ is a Banach space with respect to the norm $\|\cdot\|_\infty$ which is isometric isomorphic with $L_{b(A)}(b(E), b(F))$.

For $\xi \in E$ and $\eta \in F$ we consider the rank one homomorphism $\theta_{\eta, \xi}$ from E into F defined by $\theta_{\eta, \xi}(\zeta) = \eta \langle \xi, \zeta \rangle$. Clearly, $\theta_{\eta, \xi} \in L_A(E, F)$ and $\theta_{\eta, \xi}^* = \theta_{\xi, \eta}$. The closed linear subspace of $L_A(E, F)$ spanned by $\{\theta_{\eta, \xi}; \xi \in E, \eta \in F\}$ is denoted

by $K_A(E, F)$, and we write $K_A(E)$ for $K_A(E, E)$. Moreover, $K_A(E, F)$ may be identified with $\varprojlim_p K_{A_p}(E_p, F_p)$.

We say that the Hilbert A -modules E and F are unitarily equivalent if there is a unitary element U in $L_A(E, F)$ (namely, $U^*U = \text{id}_E$ and $UU^* = \text{id}_F$).

Given a countable family of Hilbert A -modules $\{E_n\}_n$, the set $\bigoplus_n E_n$ of all sequences $(\xi_n)_n$ with $\xi_n \in E_n$ such that $\sum_n \langle \xi_n, \xi_n \rangle$ converges in A is a Hilbert A -module with the action of A on $\bigoplus_n E_n$ defined by $(\xi_n)_n a = (\xi_n a)_n$ and the inner-product defined by $\langle (\xi_n)_n, (\eta_n)_n \rangle = \sum_n \langle \xi_n, \eta_n \rangle$. For each $p \in S(A)$, the Hilbert A_p -modules $\bigoplus_n (E_n)_p$ and $\left(\bigoplus_n E_n\right)_p$ are unitarily equivalent and so the Hilbert A -modules $\bigoplus_n E_n$ and $\varprojlim_p \bigoplus_n (E_n)_p$ are unitarily equivalent. In the particular case when $E_n = A$ for any n , the Hilbert A -module $\bigoplus_n A$ is denoted by H_A .

3. MULTIPLIER MODULES

Let A be a locally C^* -algebra and let E be a Hilbert A -module. It is not difficult to check that $L_A(A, E)$ is a Hilbert $L_A(A)$ -module with the action of $L_A(A)$ on $L_A(A, E)$ defined by $t \cdot m = t \circ m$, $t \in L_A(A, E)$ and $m \in L_A(A)$ and the $L_A(A)$ -valued inner-product defined by $\langle s, t \rangle_{L_A(A)} = s^* \circ t$. Moreover, since

$$\tilde{p}_{L_A(A)}(s^* \circ s) = \tilde{p}_{L_A(A, E)}(s)^2$$

for all $s \in L_A(A, E)$ and for all $p \in S(A)$, the topology on $L_A(A, E)$ induced by the inner product coincides with the topology determined by the family of seminorms $\{\tilde{p}_{L_A(A, E)}\}_{p \in S(A)}$. Therefore $L_A(A, E)$ is a Hilbert $L_A(A)$ -module and since $L_A(A)$ can be identified with the multiplier algebra $M(A)$ of A (see, for example, [8]), $L_A(A, E)$ becomes a Hilbert $M(A)$ -module.

DEFINITION 3.1. *The Hilbert $M(A)$ -module $L_A(A, E)$ is called the multiplier module of E , and it is denoted by $M(E)$.*

DEFINITION 3.2. *The strict topology on the multiplier module $M(E)$ of E is one generated by the family of seminorms $\{\|\cdot\|_{p, a, \xi}\}_{(p, a, \xi) \in S(A) \times A \times E}$, where $\|\cdot\|_{p, a, \xi}$ is defined by $\|t\|_{p, a, \xi} = \bar{p}_E(t(a)) + p(t^*(\xi))$.*

THEOREM 3.3. *Let A be a locally C^* -algebra and let E be a Hilbert A -module.*

- (1) $\{M(E_p); (\pi_{pq}^{A, E})_*\}_{p, q \in S(A), p \geq q}$ is an inverse system of Hilbert C^* -modules.

- (2) The Hilbert $M(A)$ -modules $M(E)$ and $\varprojlim_p M(E_p)$ are unitarily equivalent.
- (3) The isomorphism of (2) identifies the strict topology on E with the topology on $\varprojlim_p M(E_p)$ obtained by taking the inverse limit for the strict topology on the $M(E_p)$.
- (4) $M(E)$ is complete with respect to the strict topology.
- (5) The map $i_E : E \rightarrow M(E)$ defined by $i_E(\xi)(a) = \xi a$, $a \in A$ embeds E as a closed submodule of $M(E)$. Moreover, if $t \in M(E)$ then $t \cdot a = i_E(t(a))$ for all $a \in A$ and $\langle t, i_E(\xi) \rangle_{M(E)} = t^*(\xi)$ for all $\xi \in E$.
- (6) The image of i_E is dense in $M(E)$ with respect to the strict topology.

Proof. 1. Let $p, q \in S(A)$ with $p \geq q$, $t, t_1, t_2 \in M(E_p)$, $b \in M(A_p)$. Then

$$\begin{aligned}
 (\pi_{pq}^{A,E})_*(t \cdot b)(\pi_q(a)) &= \sigma_{pq}^E((t \cdot b)(\pi_p(a))) = \sigma_{pq}^E(t(b\pi_p(a))) \\
 &= (\pi_{pq}^{A,E})_*(t)(\pi_{pq}(b\pi_p(a))) \\
 &= (\pi_{pq}^{A,E})_*(t)(\pi_{pq}''(b)\pi_q(a)) \\
 &= ((\pi_{pq}^{A,E})_*(t) \cdot \pi_{pq}''(b))(\pi_q(a))
 \end{aligned}$$

and

$$\begin{aligned}
 \langle (\pi_{pq}^{A,E})_*(t_1), (\pi_{pq}^{A,E})_*(t_2) \rangle_{M(E_q)}(\pi_q(a)) &= ((\pi_{pq}^{A,E})_*(t_1))^*(\sigma_{pq}^E(t_2(\pi_p(a)))) \\
 &= (\pi_{pq}^{E,A})_*(t_1^*)(\sigma_{pq}^E(t_2(\pi_p(a)))) \\
 &= \pi_{pq}((t_1^* \circ t_2)(\pi_p(a))) \\
 &= (\pi_{pq}^{A,A})_*(t_1^* \circ t_2)(\pi_q(a)) \\
 &= (\pi_{pq}^{A,A})_*(\langle t_1, t_2 \rangle_{M(E_p)})(\pi_q(a))
 \end{aligned}$$

for all $a \in A$. From these relations we deduce that $\{M(E_p); (\pi_{pq}^{A,E})_*\}_{p,q \in S(A), p \geq q}$ is an inverse system of Hilbert C^* -modules.

2. By (1) $\varprojlim_p M(E_p)$ is a Hilbert $\varprojlim_p M(A_p)$ -module, and since $\varprojlim_p M(A_p)$ can be identified with $M(A)$, we can suppose that $\varprojlim_p M(E_p)$ is a Hilbert $M(A)$ -module. The linear map $U : M(E) \rightarrow \varprojlim_p M(E_p)$ defined by $U(t) = ((\pi_p^{A,E})_*(t))_p$

is an isomorphism of locally convex spaces [Proposition 4.7, 8]. Moreover, we have

$$\begin{aligned}\langle U(t), U(t) \rangle &= \left(\langle (\pi_p^{A,E})_*(t), (\pi_p^{A,E})_*(t) \rangle_{M(A_p)} \right)_p \\ &= \left((\pi_p^{A,E})_*(t)^* (\pi_p^{A,E})_*(t) \right)_p \\ &= \left((\pi_p^{A,A})_*(t^* \circ t) \right)_p = \langle t, t \rangle_{M(A)}\end{aligned}$$

for all $t \in M(E)$. From these facts and [Proposition 3.3, 4], we deduce that U is a unitary operator from $M(E)$ to $\varprojlim_p M(E_p)$. Therefore the Hilbert modules $M(E)$ and $\varprojlim_p M(E_p)$ are unitarily equivalent.

3. We will show that the connecting maps $(\pi_{pq}^{A,E})_*$, $p, q \in S(A)$ with $p \geq q$ are strictly continuous. For this, let $p, q \in S(A)$ with $p \geq q$. From

$$\begin{aligned}\|(\pi_{pq}^{A,E})_*(t)\|_{E_q, \pi_q(a), \sigma_q^E(\xi)} &= \|(\pi_{pq}^{A,E})_*(t) (\pi_q(a))\|_{E_q} \\ &\quad + \|((\pi_{pq}^{A,E})_*(t))^* (\sigma_q^E(\xi))\|_{A_q} \\ &= \|\sigma_{pq}^E(t(\pi_p(a)))\|_{E_q} + \|\pi_{pq}(t^*(\sigma_p^E(\xi)))\|_{A_q} \\ &\leq \|t(\pi_p(a))\|_{E_p} + \|t^*(\sigma_p^E(\xi))\|_{A_p} \\ &= \|t\|_{E_p, \pi_p(a), \sigma_p^E(\xi)}\end{aligned}$$

for all $a \in A$, for all $\xi \in E$, and for all $t \in M(E_p)$, we deduce that the map $(\pi_{pq}^{A,E})_*$ is strictly continuous. Clearly, the net $\{t_i\}_{i \in I}$ converges strictly in $M(E)$ if and only if the nets $\{(\pi_p^{A,E})_*(t_i)\}_{i \in I}$, $p \in S(A)$ converge strictly in $M(E_p)$, $p \in S(A)$.

4. Since for each $p \in S(A)$, $M(E_p)$ is strictly complete, $\varprojlim_p M(E_p)$ is strictly complete, and then by (3), $M(E)$ is strictly complete.

5. Let $p \in S(A)$. The map $i_{E_p} : E_p \rightarrow M(E_p)$ defined by $i_{E_p}(\xi_p)(a_p) = \xi_p a_p$, $a_p \in A_p$ and $\xi_p \in E_p$ embeds E_p in $M(E_p)$ (see, for example, [9]). It is not difficult to check that $\sigma_{pq}^E \circ i_{E_p} = i_{E_q} \circ (\pi_{pq}^{A,E})_*$ for all $p, q \in S(A)$ with $p \geq q$. Therefore $\{i_{E_p}\}_p$ is an inverse system of isometric linear maps. Let $i_E = \varprojlim_p i_{E_p}$. Identifying E with $\varprojlim_p E_p$ and $M(E)$ with $\varprojlim_p M(E_p)$, we can suppose that i_E is a linear map from E to $M(E)$. It is not difficult to check that $i_E(\xi)(a) = \xi a$, $i_E(\xi a) = i_E(\xi) \cdot a$ and $\langle i_E(\xi), i_E(\xi) \rangle_{M(A)} = \langle \xi, \xi \rangle$ for all $a \in A$ and for all $\xi \in E$. Moreover, if $t \in M(E)$, $a \in A$ and $\xi \in E$, then

$$(t \cdot a)(c) = t(ac) = t(a)c = i_E(t(a))(c)$$

and

$$\langle t, i_E(\xi) \rangle_{M(A)}(c) = t^*(\xi c) = t^*(\xi) c = t^*(\xi)(c)$$

for all $c \in A$.

6. Let $\{e_i\}_{i \in I}$ be an approximate unit for A and let $t \in M(E)$. By (5), $\{t \cdot e_i\}_{i \in I}$ is a net in E . Let $p \in S(A)$, $a \in A$, $\xi \in E$. Then we have

$$\begin{aligned} \|t \cdot e_i - t\|_{p,a,\xi} &= \bar{p}((t \cdot e_i - t)(a)) + p((t \cdot e_i - t)^*(\xi)) \\ &= \bar{p}(t(e_i a - a)) + p(e_i t^*(\xi) - t^*(\xi)) \\ &\leq \tilde{p}(t)p(e_i a - a) + p(e_i t^*(\xi) - t^*(\xi)). \end{aligned}$$

Since $\{e_i\}_{i \in I}$ is an approximate unit for A , $p(e_i a - a) \rightarrow 0$ and $p(e_i t^*(\xi) - t^*(\xi)) \rightarrow 0$. Therefore $\{t \cdot e_i\}_{i \in I}$ converges strictly to t . \square

REMARK 3.4. *Let A be a locally C^* -algebra. Then the multiplier module $M(A)$ coincides with the Hilbert $M(A)$ -module $M(A)$.*

REMARK 3.5. *According to the statement (5) of the above theorem, E can be identified with a closed submodule of $M(E)$. Thus, the range of an element ξ under i_E will be denoted by ξ .*

REMARK 3.6. *According to the statement (5) of the above theorem, $EA \subseteq M(E)A \subseteq E$. From this fact, and taking into account that EA is dense in E , we conclude that $M(E)A$ is dense in E .*

REMARK 3.7. (1) *If A is unital, then E is complete with respect to the strictly topology and so $E = M(E)$.*
 (2) *If $K_A(E)$ is unital, then, for each $p \in S(A)$, $K_{A_p}(E_p)$ is unital and by [Proposition 2.8, 1], $M(E_p) = E_p$. From these facts and Theorem 3.3 (2) we deduce that $E = M(E)$.*

REMARK 3.8. *The map $\Phi : b(L_A(A, E)) \rightarrow L_{b(A)}(b(A), b(E))$ defined by $\Phi(t) = t|_{b(A)}$, where $t|_{b(A)}$ denotes the restrictions of t on $b(A)$, is an isometric isomorphism of Banach spaces [Theorem 3.7, 5]. Since*

$$\Phi(t \cdot b)(a) = (t \cdot b)|_{b(A)}(a) = t(ba)$$

and

$$(\Phi(t) \cdot b)(a) = (t|_E \cdot b)(a) = t(ba)$$

for all $t \in b(L_A(A, E))$, for all $b \in M(b(A))$, and for all $a \in b(A)$, Φ is a unitary operator from $b(L_A(A, E))$ to $L_{b(A)}(b(A), b(E))$ [7]. Therefore the Hilbert $M(b(A))$ -modules $b(M(E))$ and $M(b(E))$ are unitarily equivalent.

Let $\{E_n\}_n$ be a countable family of Hilbert A -modules and let

$$\text{str.-}\bigoplus_n M(E_n) = \{(t_n)_n; t_n \in M(E_n) \text{ and } \sum_n t_n^* \circ t_n \text{ converges strictly in } M(A)\}.$$

If α is a complex number and $(t_n)_n$ is an element in $\text{str.-}\bigoplus_n M(E_n)$, then clearly $(\alpha t_n)_n$ is an element in $\text{str.-}\bigoplus_n M(E_n)$.

Let $(t_n)_n \in \text{str.-}\bigoplus_n M(E_n)$, and let $t = \text{str.-}\lim_n \sum_{k=1}^n t_k^* \circ t_k$. Clearly, $\{\sum_{k=1}^n t_k^* \circ t_k\}_n$ is an increasing sequence of positive elements in $M(A)$. Then for any element a in A , and for any $p \in S(A)$, $\{p(\sum_{k=1}^n a^* t_k^*(t_k(a)))\}_n$ is an increasing sequence of positive numbers which converges to $p(a^* t(a))$. If $\{e_i\}_i$ is an approximate unit for A , then

$$\begin{aligned} \tilde{p}_{L_A(A)}(\sum_{k=1}^n t_k^* \circ t_k) &= \sup\{p(\sum_{k=1}^n t_k^*(t_k(a))); a \in A, p(a) \leq 1\} \\ &= \sup\{\lim_i p(\sum_{k=1}^n e_i t_k^*(t_k(e_i a))), a \in A, p(a) \leq 1\} \\ &\leq \lim_i p(\sum_{k=1}^n e_i t_k^*(t_k(e_i))) \\ &\leq \lim_i p(e_i t(e_i)) \leq \tilde{p}_{L_A(A, E)}(t). \end{aligned}$$

Let $(t_n)_n, (s_n)_n \in \text{str.-}\bigoplus_n M(E_n)$, $t = \text{str.-}\lim_n \sum_{k=1}^n t_k^* \circ t_k$, $s = \text{str.-}\lim_n \sum_{k=1}^n s_k^* \circ s_k$, $a \in A$ and $p \in S(A)$. Then

$$\begin{aligned}
p\left(\sum_{k=n}^m s_k^*(t_k(a))\right) &= p\left(\sum_{k=n}^m \langle s_k, t_k \rangle_{M(A)}(a)\right) = p\left(\sum_{k=n}^m \langle s_k, t_k \rangle_{M(A)} \cdot a\right) \\
&= \tilde{p}_{L(A)}\left(\sum_{k=n}^m \langle s_k, t_k \cdot a \rangle_{M(A)}\right) \\
&= \tilde{p}_{L(A)}(\langle (s_k)_{k=n}^m, (t_k \cdot a)_{k=n}^m \rangle_{M(A)}) \\
&\quad \text{Cauchy-Schwarz Inequality} \\
&\leq \tilde{p}_{L(A)}\left(\sum_{k=n}^m \langle s_k, s_k \rangle_{M(A)}\right)^{1/2} \tilde{p}_{L(A)}\left(\sum_{k=n}^m \langle t_k \cdot a, t_k \cdot a \rangle_{M(A)}\right)^{1/2} \\
&\leq \tilde{p}_{L(A)}(s)^{1/2} \tilde{p}_{L(A)}\left(\sum_{k=n}^m (t_k^* \circ t_k)(a)\right)^{1/2} p(a)^{1/2}
\end{aligned}$$

and

$$p\left(\sum_{k=n}^m t_k^*(s_k(a))\right) \leq \tilde{p}_{L(A)}(t)^{1/2} \tilde{p}_{L(A)}\left(\sum_{k=n}^m (s_k^* \circ s_k)(a)\right)^{1/2} p(a)^{1/2}$$

for all positive integers n and m with $m \geq n$. From these facts, we deduce that the sequence $\{\sum_{k=1}^n s_k^* \circ t_k\}_n$ converges strictly in $M(A)$ and then $(t_n + s_n)_n \in \text{str-}\bigoplus_n M(E_n)$, since

$$\begin{aligned}
p\left(\sum_{k=n}^m (t_k + s_k)^*((t_k + s_k)(a))\right) &\leq p\left(\sum_{k=n}^m t_k^*(t_k(a))\right) + p\left(\sum_{k=n}^m s_k^*(s_k(a))\right) \\
&\quad + p\left(\sum_{k=n}^m t_k^*(s_k(a))\right) + p\left(\sum_{k=n}^m s_k^*(t_k(a))\right)
\end{aligned}$$

for all positive integers n and m with $n \geq m$. It is not difficult to check that $\text{str-}\bigoplus_n M(E_n)$ with the addition of two elements and the multiplication of an element in $\text{str-}\bigoplus_n M(E_n)$ by a complex number defined above is a complex vector space.

Let $b \in M(A)$ and $(t_n)_n \in \text{str-}\bigoplus_n M(E_n)$. From

$$\begin{aligned}
p\left(\sum_{k=n}^m (t_k \cdot b)^*((t_k \cdot b)(a))\right) &= p\left(\sum_{k=n}^m b^* t_k^*(t_k(ba))\right) \\
&\leq p\left(b^* \sum_{k=n}^m t_k^*(t_k(ba))\right) \\
&\leq p(b)p\left(\sum_{k=n}^m t_n^*(t_n(ba))\right)
\end{aligned}$$

for all $a \in A$, for all $p \in S(A)$, and for all positive integers n and m with $m \geq n$, we conclude that $\sum_n (t_n \cdot b)^* \circ (t_n \cdot b)$ converges strictly in $M(A)$ and so $(t_n \cdot b)_n \in \text{str.-}\bigoplus_n M(E_n)$.

THEOREM 3.9. *Let $\{E_n\}_n$ be a countable family of Hilbert A -modules. Then the vector space $\text{str.-}\bigoplus_n M(E_n)$ is a Hilbert $M(A)$ -module with the module action defined by $(t_n)_n \cdot b = (t_n \cdot b)_n$ and the $M(A)$ -valued inner product defined by*

$$\langle (t_n)_n, (s_n)_n \rangle_{M(A)} = \text{str.-}\lim_n \sum_{k=1}^n t_k^* \circ s_k.$$

Moreover, the Hilbert $M(A)$ -modules $\text{str.-}\bigoplus_n M(E_n)$ and $M(\bigoplus_n E_n)$ are unitarily equivalent.

Proof. It is not difficult to check that $\text{str.-}\bigoplus_n M(E_n)$ with the action of $M(A)$ on $\text{str.-}\bigoplus_n M(E_n)$ and the inner-product defined above is a pre-Hilbert $M(A)$ -module.

Let $(t_n)_n \in \text{str.-}\bigoplus_n M(E_n)$ and $a \in A$. Then, since

$$p\left(\sum_{k=n}^m \langle t(a), t_k(a) \rangle\right) = p\left(\sum_{k=n}^m a^* t_k^*(t_k(a))\right) \leq p(a)p\left(\sum_{k=n}^m (t_k^* \circ t_k)(a)\right)$$

for all $p \in S(A)$ and for all positive integers n and m with $m \geq n$, $(t_n(a))_n \in \bigoplus_n E_n$.

It is not difficult to check that the map $U((t_n)_n)$ from A to $\bigoplus_n E_n$ defined by $U((t_n)_n)(a) = (t_n(a))_n$ is a module morphism. Let $(\xi_n)_n \in \bigoplus_n E_n$ and $p \in S(A)$.

Since

$$\begin{aligned} p\left(\sum_{k=n}^m t_k^*(\xi_k)\right) &= \sup\left\{p\left(\left\langle \sum_{k=n}^m t_k^*(\xi_k), a \right\rangle\right); p(a) \leq 1\right\} \\ &= \sup\left\{p\left(\sum_{k=n}^m \langle \xi_k, t_k(a) \rangle\right); p(a) \leq 1\right\} \\ &= \sup\left\{p(\langle (\xi_k)_{k=n}^m, (t_k(a))_{k=n}^m \rangle); p(a) \leq 1\right\} \end{aligned}$$

Cauchy-Schwarz Inequality

$$\begin{aligned} &= p\left(\sum_{k=n}^m \langle \xi_k, \xi_k \rangle\right)^{1/2} \sup\left\{p\left(\sum_{k=n}^m \langle a, t_k^*(t_k(a)) \rangle\right)^{1/2}; p(a) \leq 1\right\} \\ &= p\left(\sum_{k=n}^m \langle \xi_k, \xi_k \rangle\right)^{1/2} \sup\left\{p\left(\sum_{k=n}^m a^* t_k^*(t_k(a))\right)^{1/2}; p(a) \leq 1\right\} \\ &\leq p\left(\sum_{k=n}^m \langle \xi_k, \xi_k \rangle\right)^{1/2} \tilde{p}_{L_A(A)}\left(\sum_n t_k^* \circ t_k\right)^{1/2} \end{aligned}$$

for all positive integers n and m with $m \geq n$, $\sum_n t_n^*(\xi_n)$ converges in A . Thus we can define a linear map $U((t_n)_n)^* : \bigoplus_n M(E_n) \rightarrow A$ by

$$U((t_n)_n)^* ((\xi_n)_n) = \sum_n t_n^*(\xi_n).$$

Moreover, since

$$\begin{aligned} \langle U((t_n)_n)(a), (\xi_n)_n \rangle &= \langle (t_n(a))_n, (\xi_n)_n \rangle \\ &= \sum_n \langle t_n(a), \xi_n \rangle = \sum_n \langle a, t_n^*(\xi_n) \rangle \\ &= \langle a, U((t_n)_n)^* ((\xi_n)_n) \rangle \end{aligned}$$

for all $a \in A$ and for all $(\xi_n)_n \in \bigoplus_n E_n$, $U((t_n)_n) \in M(\bigoplus_n E_n)$. Thus, we have defined a map U from $\text{str-}\bigoplus_n M(E_n)$ to $M(\bigoplus_n E_n)$. It is not difficult to check that U is a module morphism. Moreover,

$$\begin{aligned} \langle U((t_n)_n), U((s_n)_n) \rangle_{M(A)}(a) &= U((t_n)_n)^*(U((s_n)_n)(a)) \\ &= U((t_n)_n)^*((s_n(a))_n) \\ &= \sum_n t_n^*(s_n(a)) = \langle (t_n)_n, (s_n)_n \rangle_{M(A)}(a) \end{aligned}$$

for all $a \in A$ and for all $(t_n)_n, (s_n)_n \in \text{str-}\bigoplus_n M(E_n)$.

Now, we will show that U is surjective. Let m be a positive integer. Clearly, the map $P_m : \bigoplus_n E_n \rightarrow E_m$ defined by $P_m((\xi_n)_n) = \xi_m$ is an element in $L_A(\bigoplus_n E_n, E_m)$. Moreover, P_m^* is the embedding of E_m in $\bigoplus_n E_n$.

Let $t \in M(\bigoplus_n E_n)$, and $t_n = P_n \circ t$ for each positive integer n . Then $t_n \in M(E_n)$ for each positive integer n and $t(a) = (t_n(a))_n$ for all $a \in A$. Therefore $\sum_n a^* t_n^*(t_n(a))$ converges in A for all $a \in A$. Moreover, $\sum_n a^* t_n^*(t_n(a)) = a^* t^*(t(a))$ for all $a \in A$, and then

$$\begin{aligned} \tilde{p}_{L_A(A)} \left(\sum_{k=n}^m t_k^* \circ t_k \right) &= \sup \left\{ p \left(\left\langle \left(\sum_{k=n}^m t_k^* \circ t_k \right) (a), a \right\rangle \right); p(a) \leq 1 \right\} \\ &= \sup \left\{ p \left(\sum_{k=n}^m a^* t_k^*(t_k(a)) \right); p(a) \leq 1 \right\} \\ &\leq \sup \{ p(a^* t^*(t(a))); p(a) \leq 1 \} \leq \tilde{p}_{L_A(A)}(t^* \circ t) \end{aligned}$$

for all positive integers n and m with $m \geq n$ and for all $p \in S(A)$.

Let $a \in A$. From

$$\begin{aligned}
p \left(\sum_{k=n}^m t_k^* (t_k(a)) \right)^2 &= p \left(\left\langle \sum_{k=n}^m t_k^* (t_k(a)), \sum_{k=n}^m t_k^* (t_k(a)) \right\rangle \right) \\
&= p \left(\left\langle \left(\sum_{k=n}^m t_k^* \circ t_k \right) (a), \left(\sum_{k=n}^m t_k^* \circ t_k \right) (a) \right\rangle \right) \\
&= \left\| \left\langle (\pi_p^{A,A})_* \left(\sum_{k=n}^m t_k^* \circ t_k \right) (\pi_p(a)), (\pi_p^{A,A})_* \left(\sum_{k=n}^m t_k^* \circ t_k \right) (\pi_p(a)) \right\rangle \right\|_{A_p} \\
&\leq \left\| (\pi_p^{A,A})_* \left(\sum_{k=n}^m t_k^* \circ t_k \right) \right\|_{L_{A_p}(A_p)} \left\| \left\langle \pi_p(a), (\pi_p^{A,A})_* \left(\sum_{k=n}^m t_k^* \circ t_k \right) (\pi_p(a)) \right\rangle \right\|_{A_p} \\
&\leq \tilde{p}_{L_A(A)} \left(\sum_{k=n}^m t_k^* \circ t_k \right) p \left(\left\langle a, \left(\sum_{k=n}^m t_k^* \circ t_k \right) (a) \right\rangle \right) \\
&= \tilde{p}_{L_A(A)} (t^* \circ t) p \left(\sum_{k=n}^m a^* t_k^* (t_k(a)) \right)
\end{aligned}$$

for all positive integers n and m with $m \geq n$ and for all $p \in S(A)$, we conclude that $\sum_n t_n^* (t_n(a))$ converges in A . Therefore $(t_n)_n \in \text{str.-}\bigoplus_n M(E_n)$. Moreover, $U((t_n)_n) = t$ and so U is surjective. From this fact and taking into that

$$\langle U((t_n)_n), U((t_n)_n) \rangle_{M(A)} = \langle (t_n)_n, (t_n)_n \rangle$$

for all $(t_n)_n \in \text{str.-}\bigoplus_n M(E_n)$, we conclude that $\text{str.-}\bigoplus_n M(E_n)$ is a Hilbert $M(A)$ -module and moreover, U is a unitary operator [Proposition 3.3, 4]. Therefore the Hilbert $M(A)$ -modules $\text{str.-}\bigoplus_n M(E_n)$ and $M(\bigoplus_n E_n)$ are unitarily equivalent and the proposition is proved. \square

REMARK 3.10. Let $\{E_n\}_n$ be a countable family of Hilbert A -modules. In general, $\bigoplus_n M(E_n)$ is a submodule of $M(\bigoplus_n E_n)$.

REMARK 3.11. If A is unital, then $\bigoplus_n M(E_n) = M(\bigoplus_n E_n)$.

4. OPERATORS ON MULTIPLIER MODULES

Let E and F be two Hilbert A -modules. If $T \in L_{M(A)}(M(E), M(F))$, then

$$T(E) \subseteq \overline{T(M(E)A)} = \overline{T(M(E))A} \subseteq \overline{M(F)A} = F.$$

Therefore $T(E) \subseteq F$. Clearly $T|_E$, the restriction of T on E , is a module morphism. Moreover, $T|_E \in L(E, F)$, since

$$\begin{aligned}
\langle T|_E(\xi), \eta \rangle &= \langle T(i_E(\xi)), i_E(\eta) \rangle_{M(A)} \\
&= \langle i_E(\xi), T^*(i_E(\eta)) \rangle_{M(A)} = \langle \xi, T^*|_F(\eta) \rangle
\end{aligned}$$

for all $\xi \in E$ and for all $\eta \in F$.

THEOREM 4.1. *Let E and F be two Hilbert A -modules.*

- (1) *Then the complete locally convex spaces $L_{M(A)}(M(E), M(F))$ and $L_A(E, F)$ are isomorphic.*
- (2) *The locally C^* -algebras $L_{M(A)}(M(E))$ and $L_A(E)$ are isomorphic.*

Proof. 1. We show that the map $\Phi : L_{M(A)}(M(E), M(F)) \rightarrow L_A(E, F)$ defined by $\Phi(T) = T|_E$ is an isomorphism of locally convex spaces. Clearly, Φ is a linear map. Moreover, Φ is continuous, since

$$\tilde{p}_{L_A(E, F)}(\Phi(T)) = \tilde{p}_{L_A(E, F)}(T|_E) \leq \tilde{p}_{L_{M(A)}(M(E), M(F))}(T)$$

for all $T \in L_{M(A)}(M(E), M(F))$ and for all $p \in S(A)$. To show that Φ is injective, let $T \in L_{M(A)}(M(E), M(F))$ such that $T|_E = 0$. Then

$$\begin{aligned} \bar{p}_{M(F)}(T(s)) &= \sup\{\bar{p}_F(T(s)(a)); p(a) \leq 1\} \\ &= \sup\{\bar{p}_F(T(s \cdot a)); p(a) \leq 1\} = 0 \end{aligned}$$

for all $s \in M(E)$ and for all $p \in S(A)$. Therefore $T = 0$.

Let $T \in L(E, F)$. Then, for each $s \in M(E)$, $T \circ s \in M(F)$. Define $\tilde{T} : M(E) \rightarrow M(F)$ by $\tilde{T}(s) = T \circ s$. Clearly, \tilde{T} is linear. Moreover,

$$\tilde{T}(s \cdot b)(a) = T((s \cdot b)(a)) = T(s(ba)) = \tilde{T}(s)(ba) = (\tilde{T}(s) \cdot b)(a)$$

and

$$\langle \tilde{T}(s), r \rangle_{M(A)} = s^* \circ T^* \circ r = \langle s, T^* \circ r \rangle_{M(A)}$$

for all $s \in M(E)$, for all $r \in M(F)$, for all $b \in M(A)$, and for all $a \in A$. From these relations we conclude that \tilde{T} is an adjointable module morphism. Therefore $\tilde{T} \in L_{M(A)}(M(E), M(F))$. It is not difficult to check that $\tilde{T}|_E = T$. Thus we showed that Φ is surjective. Therefore Φ is a continuous bijective linear map from $L_{M(A)}(M(E), M(F))$ to $L_A(E, F)$. Moreover, $\Phi^{-1}(T)(s) = T \circ s$ for all $s \in M(E)$ and for all $T \in L_A(E, F)$. To show that Φ is an isomorphism of locally convex spaces it remains to prove that Φ^{-1} is continuous. Let $p \in S(A)$ and $T \in L_A(E, F)$. Then

$$\begin{aligned} \tilde{p}_{L_{M(A)}(M(E), M(F))}(\Phi^{-1}(T)) &= \sup\{\bar{p}_{M(F)}(T \circ s); \bar{p}_{M(E)}(s) \leq 1\} \\ &\leq \sup\{\tilde{p}_{L_A(E, F)}(T)\tilde{p}_{L_A(A, E)}(s); \bar{p}_{M(E)}(s) \leq 1\} \\ &\leq \tilde{p}_{L_A(E, F)}(T). \end{aligned}$$

Therefore Φ^{-1} is continuous. Moreover, we showed that $\tilde{p}_{L_{M(A)}(M(E), M(F))}(T) = \tilde{p}_{L_A(E, F)}(T|_E)$ for all $p \in S(A)$.

2. By (1) we deduce that the map $\Phi : L_{M(A)}(M(E)) \rightarrow L_A(E)$ defined by $\Phi(T) = T|_E$ is an isomorphism of locally convex spaces. It is not difficult to check that $\Phi(T_1 T_2) = \Phi(T_1) \Phi(T_2)$ and $\Phi(T^*) = \Phi(T)^*$ for all $T, T_1, T_2 \in L_{M(A)}(M(E))$. Therefore Φ is an isomorphism of locally C^* -algebras. \square

If E and F are two unitarily equivalent full Hilbert C^* -modules, then the Hilbert C^* -modules $M(E)$ and $M(F)$ are unitarily equivalent [Proposition 1.7, 1]. This result is also valid in the context of Hilbert modules over locally C^* -algebras.

COROLLARY 4.2. *Let E and F be two Hilbert A -modules. Then E and F are unitarily equivalent if and only if $M(E)$ and $M(F)$ are unitarily equivalent.*

Proof. Indeed, two Hilbert A -modules E and F are unitarily equivalent if and only if there is a unitary operator U in $L(E, F)$. But, it is not difficult to check that an element $T \in L_{M(A)}(M(E), M(F))$ is unitary if and only if the restriction $T|_E$ of T on E is a unitary operator in $L(E, F)$. Therefore, the Hilbert A -modules E and F are unitarily equivalent if and only if the Hilbert $M(A)$ -modules $M(E)$ and $M(F)$ are unitarily equivalent. \square

COROLLARY 4.3. *If E is a Hilbert A -module, then $K_A(E)$ is isomorphic with an essential ideal of $K_{M(A)}(M(E))$.*

Proof. By the proof of Theorem 4.1, $\Phi^{-1}(K_A(E))$ is a locally C^* -subalgebra of $L_{M(A)}(M(E))$. Moreover, the locally C^* -algebras $K_A(E)$ and $\Phi^{-1}(K_A(E))$ are isomorphic. Clearly, $\Phi^{-1}(K_A(E))$ is a two-sided $*$ -ideal of $K_{M(A)}(M(E))$. To show that $\Phi^{-1}(K_A(E))$ is essential, let $\xi, \eta \in E$. If $\Phi^{-1}(\theta_{\xi, \eta})\theta_{t_1, t_2} = 0$ for all $t_1, t_2 \in M(E)$, then

$$\theta_{\xi, \eta}((t_1 \circ t_2^* \circ t_3)(a)) = 0$$

for all $a \in A$ and for all $t_1, t_2, t_3 \in M(E)$. From this fact and taking into account that $M(E) \langle M(E), M(E) \rangle_{M(A)} A$ is dense in E , we conclude that $\theta_{\xi, \eta} = 0$. \square

REMARK 4.4. *If $T \in L_{M(A)}(M(E), M(F))$, then T is strictly continuous. Indeed, if $\{s_i\}_{i \in I}$ is a net in $M(E)$ which converges strictly to 0, then from*

$$\bar{p}_F(T(s_i)(a)) = p_F(T(s_i \cdot a)) = \bar{p}_F(T|_E(s_i(a))) \leq \tilde{p}_{L_A(E, F)}(T|_E) \bar{p}_E(s_i(a))$$

and

$$p(T(s_i)^*(\xi)) = p(s_i^*(T^*(\xi)))$$

for all $p \in S(A)$, for all $a \in A$, for all $\xi \in F$ and for all $i \in I$, we conclude that the net $\{T(s_i)\}_{i \in I}$ converges strictly to 0.

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